第 1 章　極限與函數

1-1　數列及其極限

1.　判斷下列各無窮數列為收斂或發散數列，若為收斂數列，求其極限．

(1) $\left\langle \dfrac{n-1}{n^2} \right\rangle$　(2) $\left\langle \dfrac{1-n^2}{n} \right\rangle$　(3) $\left\langle \dfrac{2n}{1-3n} \right\rangle$　(4) $\left\langle (-1.01)^n \right\rangle$　(5) $\left\langle 1+0.9^n \right\rangle$

解

(1) $\lim\limits_{n\to\infty} \dfrac{n-1}{n^2} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}-\frac{1}{n^2}}{1} = 0$ ．

(2) $\lim\limits_{n\to\infty} \dfrac{1-n^2}{n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}-n}{1}$ ，

因為當 n 趨向無限大時，$\left\langle \dfrac{1}{n}-n \right\rangle$ 會趨向負無限大，

所以 $\left\langle \dfrac{1-n^2}{n} \right\rangle$ 為發散數列．

(3) $\lim\limits_{n\to\infty} \dfrac{2n}{1-3n} = \lim\limits_{n\to\infty} \dfrac{2}{\frac{1}{n}-3} = -\dfrac{2}{3}$ ．

(4) 因為 $-1.01 < -1$，所以數列 $\left\langle (-1.01)^n \right\rangle$ 為發散數列．

(5) 因為 $-1 < 0.9 \le 1$，所以數列 $\left\langle 0.9^n \right\rangle$ 為收斂數列，且 $\lim\limits_{n\to\infty} (0.9)^n = 0$ ．

即 $\lim\limits_{n\to\infty} \left(1+0.9^n\right) = 1 + \lim\limits_{n\to\infty} (0.9)^n = 1$ ．

2. 求下列各無窮數列的極限：

(1) $\left\langle \dfrac{n^2+1}{n^2-1} \right\rangle$ (2) $\left\langle \dfrac{999+n}{999-n} \right\rangle$ (3) $\left\langle \dfrac{n-n^2}{n+n^2} \right\rangle$ (4) $\left\langle \dfrac{2+3n+4n^2}{n^3-1} \right\rangle$

解

(1) $\lim\limits_{n\to\infty} \dfrac{n^2+1}{n^2-1} = \lim\limits_{n\to\infty} \dfrac{1+\dfrac{1}{n^2}}{1-\dfrac{1}{n^2}} = 1$.

(2) $\lim\limits_{n\to\infty} \dfrac{999+n}{999-n} = \lim\limits_{n\to\infty} \dfrac{\dfrac{999}{n}+1}{\dfrac{999}{n}-1} = -1$.

(3) $\lim\limits_{n\to\infty} \dfrac{n-n^2}{n+n^2} = \lim\limits_{n\to\infty} \dfrac{\dfrac{1}{n}-1}{\dfrac{1}{n}+1} = -1$.

(4) $\lim\limits_{n\to\infty} \dfrac{2+3n+4n^2}{n^3-1} = \lim\limits_{n\to\infty} \dfrac{\dfrac{2}{n^3}+\dfrac{3}{n^2}+\dfrac{4}{n}}{1-\dfrac{1}{n^3}} = \dfrac{0}{1} = 0$.

3. 求下列各無窮數列的極限：

(1) $\lim\limits_{n\to\infty}\left(\dfrac{2n-1}{n}-\dfrac{n}{n-2}\right)$ (2) $\lim\limits_{n\to\infty}\left(\dfrac{n^2+1}{n-1}-\dfrac{n^2-1}{n+1}\right)$

(3) $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+n}{n^2-1}$ (4) $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\cdots+n^2}{n^2+2n^3}$

解

(1) $\lim\limits_{n\to\infty}\left(\dfrac{2n-1}{n}-\dfrac{n}{n-2}\right) = \lim\limits_{n\to\infty}\dfrac{2n-1}{n} - \lim\limits_{n\to\infty}\dfrac{n}{n-2} = 2-1=1$.

(2) $\lim\limits_{n\to\infty}\left(\dfrac{n^2+1}{n-1}-\dfrac{n^2-1}{n+1}\right) = \lim\limits_{n\to\infty}\dfrac{n^3+n^2+n+1-n^3+n^2+n-1}{n^2-1} = \lim\limits_{n\to\infty}\dfrac{2n^2+2n}{n^2-1} = 2$.

(3) $\lim\limits_{n\to\infty}\dfrac{1+2+3+\cdots+n}{n^2-1} = \lim\limits_{n\to\infty}\dfrac{\dfrac{n(n+1)}{2}}{(n-1)(n+1)} = \lim\limits_{n\to\infty}\dfrac{n}{2(n-1)} = \dfrac{1}{2}$.

(4) $\lim\limits_{n\to\infty}\dfrac{1^2+2^2+\cdots+n^2}{n^2+2n^3} = \lim\limits_{n\to\infty}\dfrac{\dfrac{n(n+1)(2n+1)}{6}}{n^2(2n+1)} = \lim\limits_{n\to\infty}\dfrac{n+1}{6n} = \dfrac{1}{6}$.

4. 求下列各無窮數列的極限：

(1) $\displaystyle\lim_{n\to\infty}\frac{3^n+2^n}{5^n}$　　(2) $\displaystyle\lim_{n\to\infty}\frac{3^n-2^{n+1}}{3^{n+1}+2^n}$

(3) $\displaystyle\lim_{n\to\infty}\left(\frac{-3n^2+n}{2n^2+1}\right)\left(\frac{3n+4}{n-1}\right)$　　(4) $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$

解

(1) $\displaystyle\lim_{n\to\infty}\frac{3^n+2^n}{5^n}=\lim_{n\to\infty}\left(\frac{3^n}{5^n}+\frac{2^n}{5^n}\right)=\lim_{n\to\infty}\left[\left(\frac{3}{5}\right)^n+\left(\frac{2}{5}\right)^n\right]=\lim_{n\to\infty}\left(\frac{3}{5}\right)^n+\lim_{n\to\infty}\left(\frac{2}{5}\right)^n=0$.

(2) 將 $\dfrac{3^n-2^{n+1}}{3^{n+1}+2^n}$ 分子分母同除以 3^n 得到 $\dfrac{1-2\cdot\left(\frac{2}{3}\right)^n}{3+\left(\frac{2}{3}\right)^n}$，

因此 $\displaystyle\lim_{n\to\infty}\frac{3^n-2^{n+1}}{3^{n+1}+2^n}=\lim_{n\to\infty}\frac{1-2\cdot\left(\frac{2}{3}\right)^n}{3+\left(\frac{2}{3}\right)^n}=\frac{\lim_{n\to\infty}\left[1-2\left(\frac{2}{3}\right)^n\right]}{\lim_{n\to\infty}\left[3+\left(\frac{2}{3}\right)^n\right]}=\frac{1-2\cdot 0}{3+0}=\frac{1}{3}$.

(3) 因為 $\displaystyle\lim_{n\to\infty}\left(\frac{-3n^2+n}{2n^2+1}\right)=\lim_{n\to\infty}\left(\frac{-3+\frac{1}{n}}{2+\frac{1}{n^2}}\right)=-\frac{3}{2}$，$\displaystyle\lim_{n\to\infty}\left(\frac{3n+4}{n-1}\right)=\lim_{n\to\infty}\left(\frac{3+\frac{4}{n}}{1-\frac{1}{n}}\right)=3$，

所以 $\displaystyle\lim_{n\to\infty}\left(\frac{-3n^2+n}{2n^2+1}\right)\left(\frac{3n+4}{n-1}\right)=\lim_{n\to\infty}\left(\frac{-3n^2+n}{2n^2+1}\right)\cdot\lim_{n\to\infty}\left(\frac{3n+4}{n-1}\right)=\left(-\frac{3}{2}\right)\cdot 3=-\frac{9}{2}$.

(4) 因為 $\displaystyle\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)$

$\displaystyle=\left[\left(1-\frac{1}{2}\right)\left(1+\frac{1}{2}\right)\right]\left[\left(1-\frac{1}{3}\right)\left(1+\frac{1}{3}\right)\right]\left[\left(1-\frac{1}{4}\right)\left(1+\frac{1}{4}\right)\right]\cdots\left[\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right]$

$\displaystyle=\left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\left(\frac{3}{4}\cdot\frac{5}{4}\right)\cdots\left(\frac{n-1}{n}\cdot\frac{n+1}{n}\right)$

$\displaystyle=\frac{n+1}{2n}$，

所以 $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\lim_{n\to\infty}\frac{n+1}{2n}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}=\frac{1}{2}$.

5. (1)已知 $\lim\limits_{n\to\infty}\dfrac{an^2+bn+4}{2n+3}=3$，求常數 a， b .

(2)已知 $\lim\limits_{n\to\infty}\left(\dfrac{n^2+1}{n+1}-an-b\right)=2$，求常數 a， b .

解

(1) 因為 $\lim\limits_{n\to\infty}\dfrac{an^2+bn+4}{2n+3}=\lim\limits_{n\to\infty}\dfrac{an+b+\dfrac{4}{n}}{2+\dfrac{3}{n}}=3$，所以 $a=0$，而 $\dfrac{b}{2}=3$，即 $b=6$.

(2)
$$\lim_{n\to\infty}\left(\frac{n^2+1}{n+1}-an-b\right)=\lim_{n\to\infty}\left(\frac{n^2+1-an^2-an-bn-b}{n+1}\right)$$
$$=\lim_{n\to\infty}\frac{(1-a)n^2-(a+b)n+(1-b)}{n+1}$$
$$=\lim_{n\to\infty}\frac{(1-a)n-(a+b)+\dfrac{(1-b)}{n}}{1+\dfrac{1}{n}}=2 .$$

因此， $1-a=0$， $-(a+b)=2$，解得 $a=1$， $b=-3$.

6. 已知 $\lim\limits_{n\to\infty}\dfrac{3n-a_n}{3n-2}=2$，且 $\lim\limits_{n\to\infty}\dfrac{a_n}{n}$ 存在，求 $\lim\limits_{n\to\infty}\dfrac{a_n}{n}$.

解 因為 $\lim\limits_{n\to\infty}\dfrac{3n-a_n}{3n-2}=2$，且 $\lim\limits_{n\to\infty}\dfrac{a_n}{n}$ 存在，將 $\lim\limits_{n\to\infty}\dfrac{3n-a_n}{3n-2}=2$變形為

$$\lim_{n\to\infty}\frac{3n-a_n}{3n-2}=\lim_{n\to\infty}\frac{3-\dfrac{a_n}{n}}{3-\dfrac{2}{n}}=\frac{\lim\limits_{n\to\infty}\left(3-\dfrac{a_n}{n}\right)}{3}=\frac{3-\lim\limits_{n\to\infty}\dfrac{a_n}{n}}{3}=2,$$

因此可得 $3-\lim\limits_{n\to\infty}\dfrac{a_n}{n}=6$，所以 $\lim\limits_{n\to\infty}\dfrac{a_n}{n}=-3$.

7.　若數列 $\dfrac{1}{3n+5}$，$\dfrac{5}{3n+5}$，$\dfrac{9}{3n+5}$，$\cdots$，$\dfrac{4n-3}{3n+5}$ 為等差數列，且這 n 項的算術平均數為 a_n，則 $\lim\limits_{n\to\infty} a_n=$______．

解　$a_n=\dfrac{1}{n}\cdot\left(\dfrac{1}{3n+5}+\dfrac{5}{3n+5}+\dfrac{9}{3n+5}+\cdots+\dfrac{4n-3}{3n+5}\right)=\dfrac{1+5+9+\cdots+(4n-3)}{3n^2+5n}$

$$=\frac{\dfrac{n(1+4n-3)}{2}}{3n^2+5n}=\frac{2n^2-n}{3n^2+5n},$$

所求 $\lim\limits_{n\to\infty} a_n=\lim\limits_{n\to\infty}\dfrac{2n^2-n}{3n^2+5n}=\lim\limits_{n\to\infty}\dfrac{2-\dfrac{1}{n}}{3+\dfrac{5}{n}}=\dfrac{2}{3}$ ．

8.　試證：當正整數 $n\geq 2$ 時，不等式 $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\cdots+\dfrac{1}{n+1}>\dfrac{n}{n+1}$ 恆成立．

解　(1) 當 $n=2$ 時，左式 $=\dfrac{1}{2}+\dfrac{1}{3}=\dfrac{5}{6}$，右式 $=\dfrac{2}{3}$，因此 $\dfrac{5}{6}>\dfrac{2}{3}$，原式成立．

(2) 設 $n=k$（$k\geq 2$）時原式成立，即 $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\cdots+\dfrac{1}{k+1}>\dfrac{k}{k+1}$，則

當 $n=k+1$ 時，

$$\text{左式}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{k+1}+\frac{1}{k+2}>\frac{k}{k+1}+\frac{1}{k+2},$$

$$\text{右式}=\frac{k+1}{k+2},$$

將兩式相減，得

$$\text{左式}-\text{右式}=\frac{k}{k+1}+\frac{1}{k+2}-\frac{k+1}{k+2}=\frac{k}{(k+1)(k+2)}>0$$

因此左式大於右式，即 $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\cdots+\dfrac{1}{k+1}+\dfrac{1}{k+2}>\dfrac{k+1}{k+2}$，

所以原式在 $n=k+1$ 時也成立．

故由數學歸納法可知：

當正整數 $n\geq 2$ 時，不等式 $\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\cdots+\dfrac{1}{n+1}>\dfrac{n}{n+1}$ 恆成立．