

第3章 指數與對數函數

3-1 指數

重點一 正整數指數與指數律

例題 1

計算下列各式的值：

$$(1) \frac{(2^2 \times 3^2) \times (2^5 \times 3^2 \times 5^3)}{(2^4 \times 3^3) \times (3 \times 5^2)} = \underline{\hspace{2cm}} \circ (5 \text{ 分})$$

$$(2) \left(\frac{5}{2}\right)^3 \times \left(\frac{6}{5}\right)^4 \times \left(\frac{4}{3}\right)^2 = \underline{\hspace{2cm}} \circ (5 \text{ 分})$$

解 (1) 原式化為 $\frac{(2^2 \times 2^5) \times (3^2 \times 3^2) \times 5^3}{2^4 \times (3^3 \times 3) \times 5^2} = \frac{2^7 \times 3^4 \times 5^3}{2^4 \times 3^4 \times 5^2} = 2^3 \times 5 = 8 \times 5 = 40$

$$(2) \text{原式化為 } \frac{5^3}{2^3} \times \frac{(2 \times 3)^4}{5^4} \times \frac{(2^2)^2}{3^2} = \frac{5^3 \times 2^4 \times 3^4 \times 2^4}{2^3 \times 5^4 \times 3^2} = \frac{2^8 \times 3^4 \times 5^3}{2^3 \times 3^2 \times 5^4}$$

$$= \frac{2^{8-3} \times 3^{4-2}}{5^{4-3}} = \frac{2^5 \times 3^2}{5^1} = \frac{32 \times 9}{5} = \frac{288}{5}$$

重點二 整數指數與指數律

例題 2

計算下列各式的值：

$$(1) 2^{-3} = \underline{\hspace{2cm}} \circ (3 \text{ 分})$$

$$(2) \left(\frac{1}{3}\right)^{-2} = \underline{\hspace{2cm}} \circ (3 \text{ 分})$$

$$(3) (-12)^{-3} = \underline{\hspace{2cm}} \circ (3 \text{ 分})$$

$$(4) 2011^0 = \underline{\hspace{2cm}} \circ (3 \text{ 分})$$

解 (1) $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

$$(2) \left(\frac{1}{3}\right)^{-2} = \frac{1}{\frac{1}{3} \times \frac{1}{3}} = \frac{1}{\frac{1}{9}} = 9$$

$$(3) (-12)^{-3} = \frac{1}{(-12)^3} = \frac{1}{-1728} = -\frac{1}{1728}$$

$$(4) 2011^0 = 1$$

例題 3

化簡下列二式：

$$(1) (a^{-3})^2 \cdot a^4 = \underline{\hspace{2cm}} \circ (4 \text{ 分})$$

$$(2) [a^3 \cdot (a^{-2})^2]^{-3} = \underline{\hspace{2cm}} \circ (4 \text{ 分})$$

解 (1) $(a^{-3})^2 \cdot a^4 = a^{(-3) \times 2} \cdot a^4$

$$= a^{-6} \cdot a^4 = a^{-6+4}$$

$$= a^{-2}$$

$$(2) [a^3 \cdot (a^{-2})^2]^{-3} = (a^3 \cdot a^{-4})^{-3}$$

$$= (a^{-1})^{-3}$$

$$= a^3$$

例題 4

設 $a = \sqrt{2} + 1$, $b = 2 + \sqrt{2}$, 試求 $\left(\frac{a^4b}{ab^4}\right)^3 \left(\frac{a^2b^{-3}}{a^{-3}b^2}\right)^{-2} = \underline{\hspace{2cm}}$ 。(5分)

$$\begin{aligned} \text{解} \quad \left(\frac{a^4b}{ab^4}\right)^3 \left(\frac{a^2b^{-3}}{a^{-3}b^2}\right)^{-2} &= (a^3b^{-3})^3 (a^5b^{-5})^{-2} \\ &= (a^9b^{-9})(a^{-10}b^{10}) = a^{-1}b \\ &= \frac{b}{a} = \frac{2+\sqrt{2}}{\sqrt{2}+1} = \frac{(2+\sqrt{2})(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)} \\ &= 2\sqrt{2} - 2 + 2 - \sqrt{2} = \sqrt{2} \end{aligned}$$

例題 5

設 $a + a^{-1} = 4$, 其中 a 是不為零的實數, 試求下列各式的值:

(1) $a^2 + a^{-2} = \underline{\hspace{2cm}}$ 。(5分) (2) $a^3 + a^{-3} = \underline{\hspace{2cm}}$ 。(5分)

(3) $a^4 + a^{-4} = \underline{\hspace{2cm}}$ 。(5分)

解 (1) $a^2 + a^{-2} = a^2 + (a^{-1})^2 = (a + a^{-1})^2 - 2 \times a \times a^{-1} = 4^2 - 2 = 14$

(2) $a^3 + a^{-3} = a^3 + (a^{-1})^3 = (a + a^{-1})(a^2 - a \times a^{-1} + a^{-2})$
 $= 4 \times (14 - 1) = 52$

(3) $a^4 + a^{-4} = (a^2)^2 + (a^{-2})^2$
 $= (a^2 + a^{-2})^2 - 2 \times a^2 \times a^{-2}$
 $= 14^2 - 2 = 194$

重點三 有理數指數與指數律

例題 6

計算下列各有理指數的值:

(1) $8^{\frac{1}{3}} = \underline{\hspace{2cm}}$ 。(3分)

(2) $216^{\frac{2}{3}} = \underline{\hspace{2cm}}$ 。(3分)

(3) $81^{-\frac{3}{4}} = \underline{\hspace{2cm}}$ 。(3分)

(4) $0.49^{-\frac{1}{2}} = \underline{\hspace{2cm}}$ 。(3分)

解 (1) $8^{\frac{1}{3}} = (2^3)^{\frac{1}{3}} = 2^1 = 2$

(2) $216^{\frac{2}{3}} = (6^3)^{\frac{2}{3}} = 6^2 = 36$

(3) $81^{-\frac{3}{4}} = (3^4)^{-\frac{3}{4}} = 3^{-3} = \frac{1}{27}$

(4) $0.49^{-\frac{1}{2}} = \left(\frac{49}{100}\right)^{-\frac{1}{2}} = \left[\left(\frac{7}{10}\right)^2\right]^{-\frac{1}{2}} = \left(\frac{7}{10}\right)^{-1} = \frac{10}{7}$

例題 7

試求 $\left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{9}{4}\right)^{\frac{1}{2}} \times (0.25)^{-1.5} = \underline{\hspace{2cm}}$ 。(5分)

$$\begin{aligned} \text{解} \quad \left(\frac{81}{16}\right)^{-0.25} \times \left(\frac{9}{4}\right)^{\frac{1}{2}} \times (0.25)^{-1.5} &= \left[\left(\frac{3}{2}\right)^4\right]^{-\frac{1}{4}} \times \left[\left(\frac{3}{2}\right)^2\right]^{\frac{1}{2}} \times \left[\left(\frac{1}{2}\right)^2\right]^{-\frac{3}{2}} \\ &= \left(\frac{3}{2}\right)^{-1} \times \frac{3}{2} \times \left(\frac{1}{2}\right)^{-3} \\ &= \frac{2}{3} \times \frac{3}{2} \times 8 = 8 \end{aligned}$$

例題 8

若 $13^x = 32$, $52^y = 256$, 則 $\frac{5}{x} - \frac{8}{y} = \underline{\hspace{2cm}}$ 。(5 分)

解 $\because 13^x = 32 \Rightarrow 13 = 32^{\frac{1}{x}} = 2^{\frac{5}{x}} \dots\dots\dots ①$

$52^y = 256 \Rightarrow 52 = 256^{\frac{1}{y}} = 2^{\frac{8}{y}} \dots\dots\dots ②$

① 得 $\frac{1}{4} = 2^{\frac{5}{x} - \frac{8}{y}} \Rightarrow 2^{-2} = 2^{\frac{5}{x} - \frac{8}{y}}$

$\Rightarrow \frac{5}{x} - \frac{8}{y} = -2$

重點四 實數指數與指數律

例題 9

試求下列各式之值：

(1) $27^{1+\sqrt{2}} \times 9^{-\frac{3\sqrt{2}}{2}} = \underline{\hspace{2cm}}$ 。(5 分)

(2) $(2^{\sqrt{2}} \times 3^{\sqrt{2}})^{\sqrt{2}} = \underline{\hspace{2cm}}$ 。(5 分)

解 (1) $27^{1+\sqrt{2}} \times 9^{-\frac{3\sqrt{2}}{2}} = (3^3)^{1+\sqrt{2}} \times (3^2)^{-\frac{3\sqrt{2}}{2}} = 3^{3+3\sqrt{2}} \times 3^{-3\sqrt{2}}$
 $= 3^{3+3\sqrt{2}-3\sqrt{2}} = 3^3 = 27$

(2) $(2^{\sqrt{2}} \times 3^{\sqrt{2}})^{\sqrt{2}} = [(2 \times 3)^{\sqrt{2}}]^{\sqrt{2}} = 6^2 = 36$

例題 10

已知 $3^x = 2$, 試求 $9^{x+1} + 27^{-x} = \underline{\hspace{2cm}}$ 。(6 分)

解 $9^{x+1} + 27^{-x} = (3^2)^{x+1} + (3^3)^{-x} = 3^{2x+2} + 3^{-3x}$
 $= (3^x)^2 \times 9 + (3^x)^{-3} = 2^2 \times 9 + 2^{-3}$
 $= 36 + \frac{1}{8} = \frac{289}{8}$

例題 11

已知 $a^{2x} = 4$, 試求：

(1) $\frac{a^{3x} + a^{-3x}}{a^x + a^{-x}} = \underline{\hspace{2cm}}$ 。(6 分)

(2) $\frac{a^{3x} - a^{-3x}}{a^x - a^{-x}} = \underline{\hspace{2cm}}$ 。(6 分)

解 (1) $\frac{a^{3x} + a^{-3x}}{a^x + a^{-x}} = \frac{(a^x + a^{-x})(a^{2x} - 1 + a^{-2x})}{a^x + a^{-x}}$
 $= a^{2x} - 1 + a^{-2x}$
 $= 4 - 1 + \frac{1}{4} = \frac{13}{4}$

(2) $\frac{a^{3x} - a^{-3x}}{a^x - a^{-x}} = \frac{(a^x - a^{-x})(a^{2x} + 1 + a^{-2x})}{a^x - a^{-x}} = a^{2x} + 1 + a^{-2x} = 4 + 1 + \frac{1}{4}$
 $= \frac{21}{4}$